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TOPOLOGICAL SYNTHESIS OF ALL 2D MECHANISMS THROUGH ASSUR GRAPHS

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ABSTRACT

It is well known that every planar kinematical linkage can be decomposed into basic topological structures referred as Assur Groups. A new reformulation of Assur Group concept into the terminology of rigidity theory, as Assur Graphs, has yielded the development of new theorems and methods.

The paper reports on an algorithm for systematic construction of Assur Graph classes, termed fundamental Assur Graphs. From each fundamental Assur Graph it is possible to derive an infinite set of different Assur Graphs. This mapping algorithm is proved to be complete and sound, i.e., all the Assur Graphs appear in the map and each graph in the map is an Assur Graph. Once we possess the mapping of all the Assur Graphs, all valid kinematical linkage topologies can be constructed through various Assur Graph compositions.

1. INTRODUCTION

This paper introduces a systematic methodology enabling the derivation of the topologies of all planar linkages. Although this generic perspective yields all planar linkage topologies, it is done using only two operations.

The idea behind the paper is based on the works of Assur (Assur 1952), but in a way different than he had anticipated. The idea behind Assur's work was to decompose every linkage into Assur groups, based on the following theorem: for every linkage there is a unique decomposition into Assur groups. While Assur had used this theorem for analysis, the work, reported in the paper, is used for topological synthesis. The work here introduces a method to construct all the Assur groups, by only two operations based on theorems from rigidity theory. Once the Assur groups have been constructed,

different combinations between them yield diverse topologies of linkages. Thus, we now have a way to construct all the topologies of any planar linkage.

The mathematical proof underlying this work is to be found in two papers published in the rigidity theory community. The first paper was published in 2003 by Berg and Jordan, who proved that there is a set of graphs that all can be derived by using only two operations (Berg and Jordan, 2003). They called these graphs – *generic cycles*. One of the results shown in a recently published paper (Servatius et al., 2009a) is that there is a relation between Assur groups and these generic cycles.

These two works constitute the mathematical foundation underlying this paper. The Assur groups were reformulated in terms of graphs, and using two operations similar to those reported in (Berg and Jordan, 2003) enable the derivation of all the Assur groups.

Since the paper deals with engineering issues using material from rigidity theory, new definitions are introduced and Assur groups are treated and defined as *Assur graphs*, and for sake of brevity are written AGs.

Many works have been published related to the topological synthesis, some are mentioned below, while the main motivation was to develop an “atlas” or a catalog of all the possible mechanisms.

The main idea that underlies most of the works published in this field was to enumerate the different topologies of mechanisms satisfying the equation of degrees-of-freedom, such as Chebyshev-Gruebler equation using graph theory techniques to handle their topology.

In 1964, Freudenstein and Dobrjanskys (1964) enumerated mechanisms according to “groups”, such that every member of a given group has the same number of k-nary links for any k.

Crossley (1964a) developed an algorithm for the enumeration of mechanisms according to their groups and found that linkages having four, six and eight links have

one, two, and sixteen chains, respectively. For linkage with ten links he reported that there are 222 different kinematic chains (Crossley, 1964b).

Thus, for each type of mechanism, different enumeration methods have been developed, for example: for geared kinematic chains a method based on Polya's theorem was developed and revealed the existence of twenty-three geared-kinematic chains with up to five links (Buchsbaum and Freudenstein, 1970); enumeration of Epicyclic Gear Mechanisms was done by enumerating the corresponding topology representation - the canonical graph representation (Tasai, 1988); for planar two-DOF linkages, Crossley found that there are 32 non-isomorphic kinematic structures with three independent loops (Crossley, 1964); in 1984, an atlas of the graphs of the kinematic chains of mechanisms with up to six links, was reported (Mayourian and Freudenstein, 1984); comprehensive description of enumeration methods for mechanisms and gear trains appears in a book by Tsai (Tsai, 2001). It was found that the number of different mechanisms identified by different methodologies is actually not unique. For example, Woo (1967) had reported on 230 different ten-link kinematic chains, which is eight more than the number reported by Crossley, appearing above.

Another approach that was examined was a method developed in Russia, by L.V. Assur in 1916, in his dissertation "Investigation of planar mechanisms of lower pairs from the point of view of their classification and structure", reprinted in (Assur, 1952).

Although the interest and works have emerged during the years while using Assur Groups, and the idea that it is possible to construct new mechanisms from them, Assur Groups are currently used mostly in Eastern Europe and have not been used for enumeration and synthesis. Some of the reasons for that appear in (Olson et al., 1985): "*However, Assur did not devise a convenient symbolic notation, and his classification system is quite unwieldy to use, but it encouraged others to seek a symbolic notation specifically for the purpose of enumerating kinematic chains*".

It should be noted, that the above obstacle has been overcome and this paper relies on the new symbolic notation of Assur Groups, which are now termed *Assur Graphs* and currently based on theorems and methods from rigidity theory (Servatius, 1999) as appears in the previous publication of the author (Servatius et al., 2010a). Once Assur Groups have been reformulated in terms of rigidity theory, the classification theorem becomes easy to implement using works reported in the last decade.

Another reservation about Assur Groups appears in (Dobranskyj, 1966), page 11: "*The theorem stated on page 8 (about the unique decomposition) has never been proven and it seems questionable whether all mechanisms*

can be derived in such a manner". This theorem was proved in previous publication of the author (Servatius et al., 2010a), this time based on theorems from rigidity and matroid theories.

But perhaps the main disadvantage of Assur groups, as appears in (Olson et al., 1985), is that they are limited to planar mechanisms only. Preliminary results (Shai 2008; 2009) and new results appearing in section 5, indicate that this obstacle also can be overcome by relying on theorems and methods from rigidity theory.

2. OVERVIEW OF THE ASSUR GRAPH CONCEPT

Assur groups, occasionally referred as Assur Structures, are widely used in the kinematical community, particularly among Russian scientists. Leonid Assur (Assur, 1952) developed these basic structures in order to make it possible to decompose any linkage into components of zero mobility, and for each one of those, to develop special methods for analysis of locations, velocities, accelerations and other physical properties.

The concept has been reformulated for the first time in rigidity theory terminology in (Servatius et al., 2009a,b), where it was defined as a rigid graph, for which deletion of any vertex results in a non-rigid graph. Accordingly, it has been shown that an Assur Graph is a basic entity applicable to treatment not only of kinematical systems, but also static systems.

The current work employs Assur graphs as the central building block of the topological synthesis of all the 2D mechanisms. Since the paper deals only with the topology of planar linkages and all the mathematical foundation of this paper is based on graph theory, the terminology used in the paper is from graph theory and can be found in any basic textbooks on the subject, such as (Swamy and Thulasiraman, 1981). For example, joints are referred to as vertices, links as edges and structures as graphs. Moreover, to avoid other terminologies used in the rigidity theory community and not in mechanical engineering, the definitions appearing in the paper are slightly modified by giving them more physical than combinatorial meaning.

To clarify the terminology used in the paper let us define the structure depicted in Figure 1 in both terminologies. In the terminology of engineering this is a determinate truss with four rods/bars, two joints – A and B, three pinned joints connecting rods 1,2 and 4 to the ground, while each rod has its specific geometry (length, inclination angle, etc.). Therefore, in engineering terminology there is a difference between the two determinate trusses in Figure 1.

In the terminology of rigidity theory the graph in Figure 1a is a rigid graph with four edges, two inner vertices, three ground vertices, three ground edges – 1,2 and 4 and there is no notion of geometry of the elements. Thus, from the rigidity theory point of view there is no difference

between the two graphs in Figure 1.

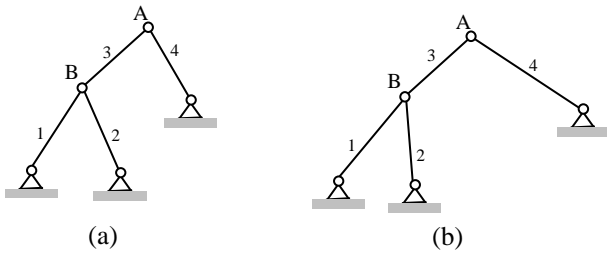


Figure 1. Two configurations with the same topology of a determinate truss (rigid graphs).

Now, we shall define Assur graphs and outline what distinguishes them from other rigid graphs.

Assur Graph – is a minimally rigid graph with $e(G)=2*v(G)$ where $e(G)$ and $v(G)$ stand for the number of edges and inner vertices of graph G , respectively. The main property of the graph is that removal of any vertex with its incident edges makes the graph non-rigid.

The graph, appearing in Figure 2(a) is an Assur Graph since the number of the edges is twice the number of the inner vertices, it is rigid and all its sub-graphs are not rigid. For example, the graph in Figure 2(b) is obtained from the graph Figure 2(a) by deleting vertex C and all its incident edges, resulting in a linkage. The system in Figure 2(c) is obtained by deleting vertex D and is also a linkage. In contrast, the structure in Figure 3(a) is not an Assur Graph since deleting vertex C results in an Assur Graph, known as the Triad, shown in Figure 3(b).

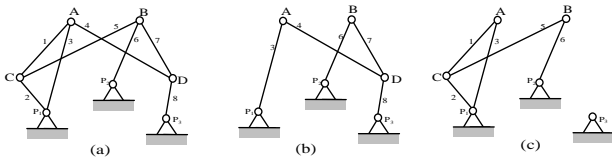


Figure 2. Example of a determinate truss that is Assur Graph.
a) Assur Graph. b,c) The graphs after deleting vertices C and D, respectively.

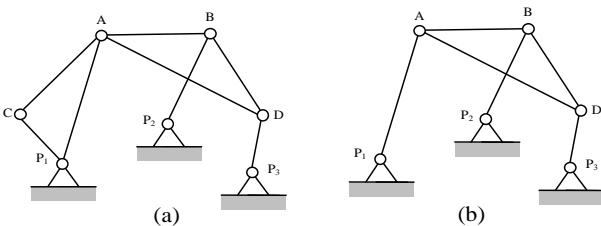


Figure 3. Example of a graph that is not an Assur Graph

In each Assur Graph there are two types of vertices: ground vertices, called also pinned vertices, and inner vertices.

For example, in a triad type Assur Graph (Figure 3b) there are three inner and ground vertices while in the dyad type Assur graph there are two ground vertices and one inner vertex.

The composition rule for constructing a determinate truss from its components (Assur Graphs) is done as follows: Let G_1 and G_2 be two Assur Graphs. G_1 is defined to be preceding G_2 if at least one ground vertex of G_1 is connected to an inner vertex of G_2 .

The decomposition process can be presented by a directed graph in which an edge $e=<u,v>$ indicates that the Assur Graph corresponding to vertex u is preceding another Assur Graph, presented by vertex v . This means that in order to decompose Assur Graph 'v', Assur Graph 'u' has first to be removed, thus this graph is termed in the paper – *decomposition graph*.

For example, in Figure 4.b the graph presents the order in which the determinate truss in Figure 4.a can be decomposed. We start with the initial vertex - a vertex to which no edge is incident. In this example the initial vertex 'F' corresponds to the dyad with the inner vertex 'F' and the two edges (F,G) and (F,J). Once this dyad is removed it is possible to remove, independently the dyads G or J, and so forth.

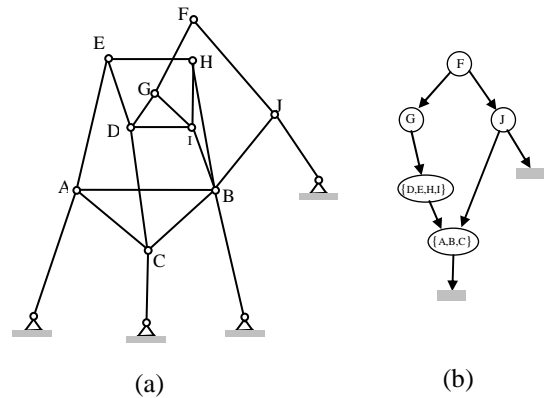


Figure 4. Example of decomposition a determinate truss into Assur Graphs.
a) The determinate truss. b) The decomposition graph.

From the above it follows that once we have all the Assur Graphs it is possible to construct all different determinate trusses by composing different Assur Graphs, each time in a different order.

The transformation from determinate trusses into planar linkages is easy and is done by just augmenting a driving

link, each time to a different ground vertex of the corresponding determinate truss. In Figure 5 we can see the three planar linkages in which the driving link is augmented, each time to a different ground vertex.

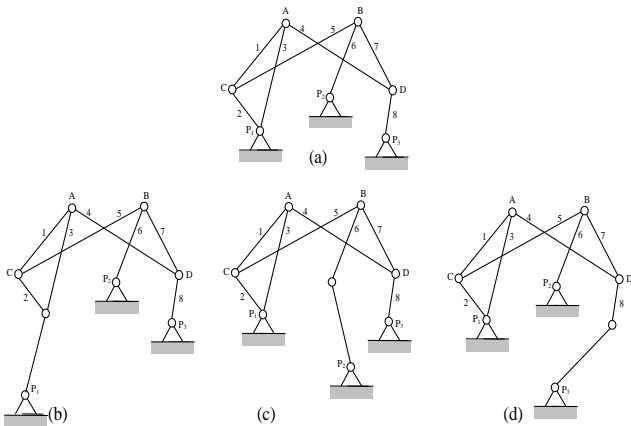


Figure 5. Example of several linkages corresponding to the same determinate truss.
 (a) The Assur Graph. (b), (c), (d) The corresponding linkages.

2. EMPLOYING ASSUR GRAPHS FOR ANALYSIS

Although the paper is aiming towards synthesis of linkages through AGs, in this section it is explained briefly that the concept of decomposition of a system into AGs enabling at each step to analyze a small component of the system. The method relies on the properties of the decomposition graph. In the first sub-section we show how to analyze mechanisms using bottom-up method, i.e., the first component to analyze corresponds to the vertex that is connected to the ground in the decomposition graph.

In the following sub-section it is shown that for analysis of a determinate truss, a top-down method is used, where the first component to analyze corresponds to the leaf vertex in the decomposition graph.

2.1 Analysis of Mechanisms through Assur Graphs

Every linkage results in a determinate truss after removing its driving link, i.e. as mentioned above, decomposed into Assur Graphs, for each of which there exist special methods for analysis.

In order to get as small matrices as possible for analysis, the analysis is done through the decomposition graph, but this time in the composition order. First, the analysis is done on the Assur Graph that all of its outer vertices are ground vertices, i.e., vertices that their velocities are known. After calculating the velocities of inner vertices of the Assur graph, this AG is then deleted and its inner vertices are replaced with ground vertices. This process

ends when all the vertices of the linkage are grounded. This analysis process is demonstrated by solving the velocities in the mechanisms appearing in Figure 6a. First, the driving link is replaced with a ground vertex and a structural scheme is constructed (Figure 6b), for which a decomposition graph with three Assur Graphs is constructed as shown in Figure 6c.

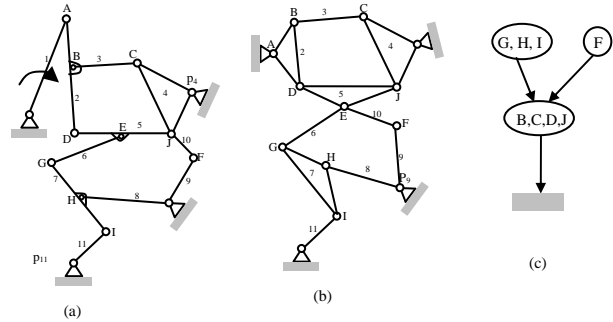


Figure 6. Example for decomposition of a linkage into Assur Graphs.
 a) The linkage. b) The structural scheme. c) The decomposition graph.

In this example, the first Assur Graph to be analyzed is the tetrad – (B,C,D,J) where the two outer vertices, A and p_4 , are ground vertices thus the inner velocities – B,C,D,E,J can be calculated. The second AG that can be analyzed is (G,H,I) or the dyad F. In this example, the second AG chosen, arbitrarily, to be analyzed is the dyad F, where now the velocity of the outer vertex – J is known from the previous AG, as shown in Figure 7b,b1. The last AG is the triad (G,H,I) whose velocity of the outer vertex E is known from the first AG – (B,C,D,J).

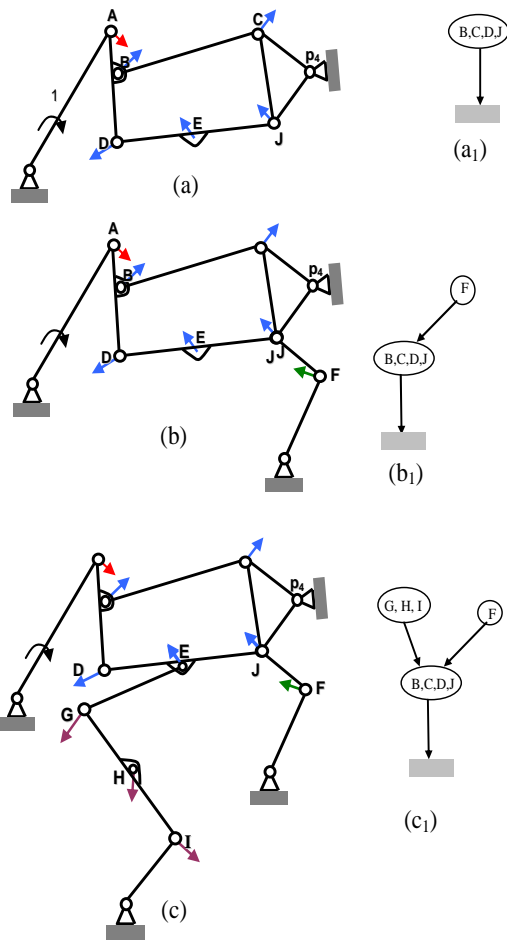


Figure 7. Analysis process of a mechanism through the decomposition into Assur Graphs.

2.2 Analysis of Determinate Trusses through Decomposition into Assur Graphs

The analysis process is done somewhat in a similar way related to the analysis of velocities in kinematics, but this time in a reversed order, i.e., in the decomposition order

The process is as follows:

1. Search for an Assur Graph that can be removed and at least one force, which is acting on one of its vertices.
2. Remove this AG, add ground vertices to its outer vertices with ground vertices and calculate the forces in its internal edges.
3. Replace all the ground edges of the removed AG with external forces with the same magnitude and direction forces in its ground edges.
4. Go to 1.

An example of applying this analysis process appears in Figures 8 and 9. The determinate truss for which the analysis is applied consists of three triads and one dyad, as

appears in Figure 8a.

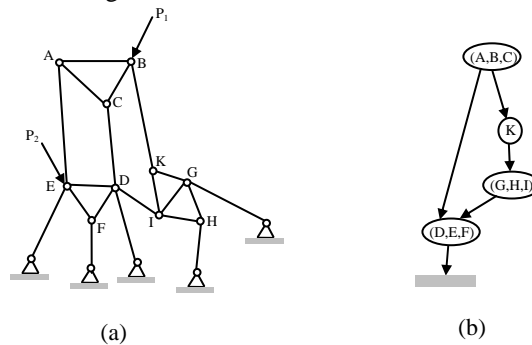


Figure 8. Determinate Truss and its corresponding decomposition graph.

Figure 9 depicts the process of analyzing the determinate truss appearing in Figure 8a, each time an AG is being analyzed. First, the triad (A,B,C) can be removed and since on one of its vertices, vertex B, acts an external force thus this AG is the first to be removed and analyzed (Figure 9b). The inner forces in the three ground edges, (AK), (CD) and (B,K), of the latter AG become external forces that act on the remaining determinate truss: P_{BK} , P_{CD} , P_{AD} , as shown in Figure 9c. This process continues and is applied on the dyad K (Figure 9c), then triad (G,H,I) as shown in Figure 9d, and ends with the analysis of the triad (E,D,F) upon which four external forces act (Figure 9e).

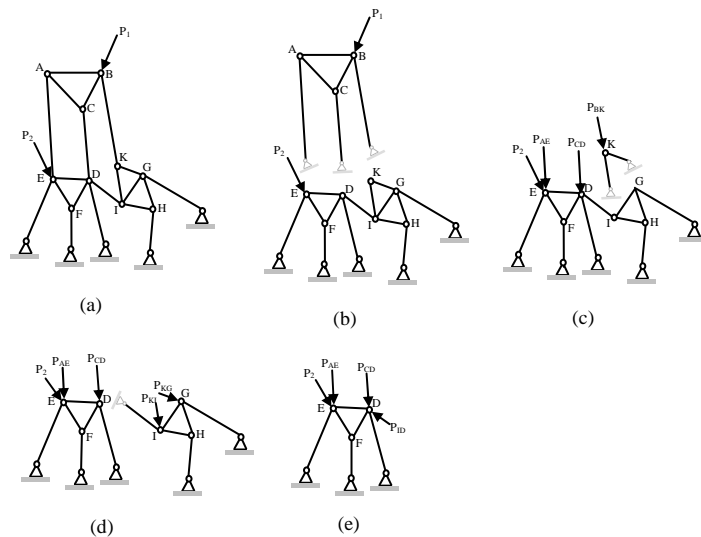


Figure 9. The process of analyzing the series of Assur Graphs composing the determinate truss appearing in Figure 8a.

3. THE PROCESS OF DERIVING ALL THE ASSUR GRAPHS IN 2D

In this section we show that it is possible to derive all the AGs in 2D by only two operations. All the AGs, although there is an infinite amount, are arranged in a very unique order as shown in the map appearing in this section. This map is proved to be complete and sound, i.e., all the AGs appear in this map and all the graphs that appear in the map are AGs.

3.1 The two extension operations

The Assur Graphs are arranged in a table with infinite rows and infinite columns; they are all derived from one basic Assur Graph, called dyad, as shown in Figure 11a, by applying only two types of extensions. The first extension, termed *fundamental extension*, produces all the Assur Graphs in the first row, called also the fundamental Assur Graphs. This operation is done by replacing a ground edge by a triangle and two new ground edges, as shown in 10(a,a1). The fundamental AGs can also be related as representatives, since from each one of them it is possible to derive an infinite number of different AGs. This is done by applying a second extension, termed *regular extension*, that divides, splits, one of the edges (x,y) by a new vertex, z , and adding a new edge (z,t) for some vertex $t=x,y$, as shown in Figure 10(b,b1).

Figure 11 depicts example of AGs that are the result of applying a sequence of extensions, starting from the basic AG – the dyad (Figure 11a). The first row presents the fundamental AGs, called also the representatives, all derived from the basic dyad through applying the fundamental extensions. For example, the fundamental AG in Figure 11b known also as Triad, is obtained by replacing the ground edge $(A,O2)$ with the triangle $\langle A,B,C \rangle$ with the two new ground edges $(C,O2)$ and $(B,O3)$. All the other infinite fundamental AGs are obtained in the same way; each time a ground edge is replaced with a triangle and two new ground edges. Now, that we can generate all the fundamental AGs, each one of them defines an infinite number of new AGs, all derived by applying successive regular extensions. For example, the AG in (b1) is derived from the fundamental AG, the triad, by applying the regular extension on edge (A,B) by adding vertex D and adding the additional edge – the ground edge – $(D,O4)$. In the subsequent extension to (b2), the additional edge is an inner edge (E,B) .

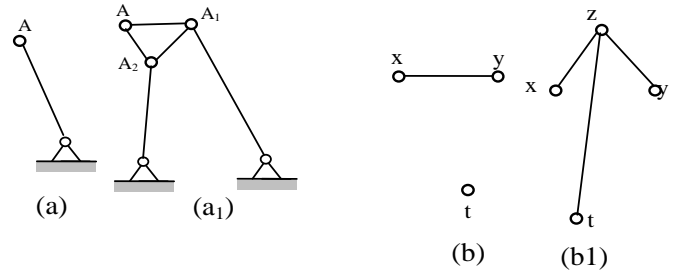


Figure 10. The result of applying fundamental and regular extensions (a1,b1) on ground and inner edges (a,b), respectively

Applying the two types of extensions infinite number of times, enables deriving the infinite map of all the AGs in 2D as shown in table 1. This work is based on theorems and methods developed in the rigidity theory community, in particular on the work of Berg and Jordan (2003) where they mathematically proved that all the rigidity circuits in 2D can be derived from the complete graph with four vertices, called K4. Based on a new relation between rigidity circuits and Assur Graphs (Servatius et al., 2010; Shai, 2008; Shai 2009) it can be proved that the following map is both complete and sound, i.e., all the AGs can be found and every graph produced by the two extensions is an AG.

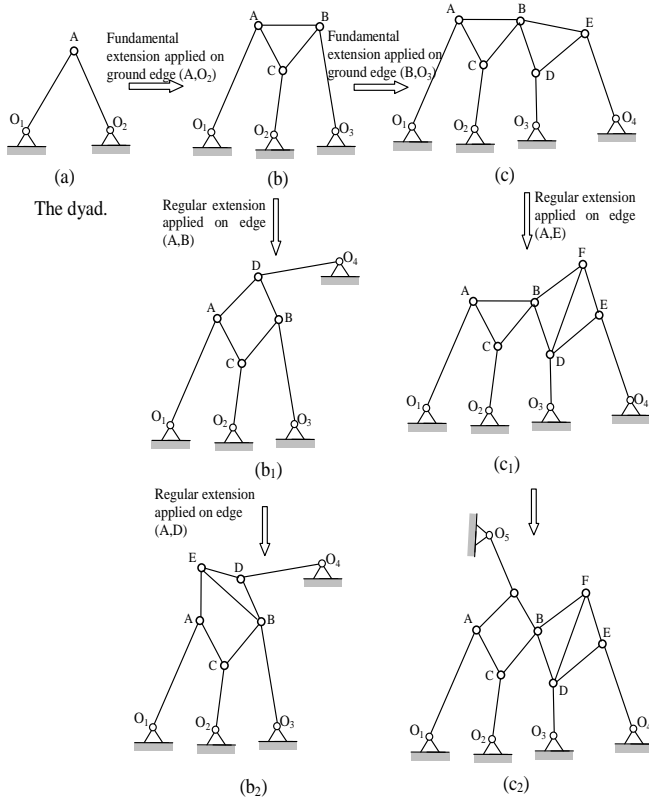


Figure 11. The resultant of AGs after applying several times the two types of extensions.

3.2 The Map of All the AGs

In the previous section we have shown that there are two types of extensions with which all new AGs can be constructed from the basic AG – the dyad. In this section we show that all the AGs can be arranged in an elegant mathematical order – in a form of an infinite map.

The first operation, the fundamental extension, produces each time an Assur Graph that is the representative of infinite AGs. Thus, each fundamental AG can be regarded as a representative of a class containing infinite AGs that can be derived only from this fundamental AG.

This canonical structure is depicted in an infinite map (table 1) where the first row corresponds to all the fundamental AGs and all the AGs in that column are those that can be derived from it. In other words, the map consists of infinite classes, each corresponding to a different column, where the element in the first row is the representative – the fundamental AG.

Class 0	Class 1	Class 2	Class 3	Class 4

Class 0	Class 1	Class 2	Class 3	Class 4

Table 1. Some classes of Assur graphs in 2D

3.3 The Proof of the Completeness and Soundness of the 2D Assur Graph Map

In this section we show that the map of all the Assur Graphs in 2D is complete and sound, i.e., all the Assur Graphs can be derived through the above two operations and all the graphs that are derived by applying the two operations are Assur Graphs. The mathematical foundation underlying this proof is based on two works reported in the rigidity theory literature. The first work was published in 2003 (Berg and Jordan, 2003) who proved that there exists an infinite set of graphs with $e=2v-2$ possessing one self-stress on all the edges, i.e., internal forces on all the edges that satisfy the force equilibrium around each vertex. Thus, in the terminology adopted in this paper, they called *floating rigid graphs*, since they are not grounded, with $e=2v-2$ that possess a unique self-stress on all the edges. They called these types of rigid graphs with the above property related to self-

stress as - *generic cycles*. Figure 12 shows floating rigid graphs with the same number of edges and vertices but only the one appearing in Figure 12b is a generic cycle since it possesses a unique self-stress. On the other hand, the floating graph in Figure 12a possesses a self-stress only in the left part of the graph and not on all edges thus it is not a generic cycle.

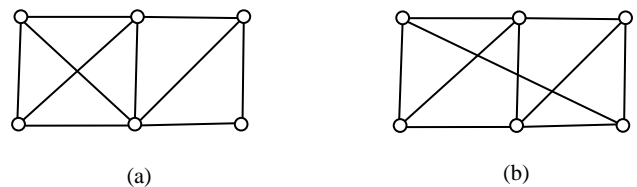


Figure 12. Example of two floating rigid graphs.
a) Not a generic cycle. b) Generic cycle

In their work, Berg and Jordan proved that it is possible to derive all the generic cycles from only one floating rigid graph, termed K4, a graph with four vertices, six edges and an edge between any two vertices as shown in Figure 13a. They have used two operations: the 1-extension and the 2-sum. The first operation is used in the current paper as well, while the latter is similar to the logical exclusive OR - XOR operation: two generic cycles are joined, glued, by deleting a common edge and the remaining edges constitute the result of the 2-sum. Example of such joining, gluing, two floating graphs K4s appears in Figure 13 where the gluing is done on edges 2 and 7, thus they do not appear in the resulted generic cycle (Figure 13.c).

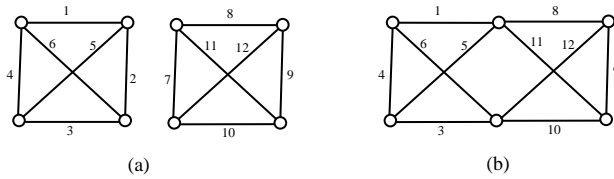


Figure 13. The 2-sum of two floating rigid graphs - K4.
a) The two floating rigid graphs- K4. b) The resultant floating rigid graph.

The second work that the proof is based on is in (Servatius et al., 2009a,b) which established the relation between Assur Graphs and generic cycles. In the latter paper it was proved that every Assur Graph corresponds to a generic cycle by contracting all of its ground vertices into one vertex. This latter graph is termed a *contracted Assur Graph*. In Figure 14a appears a triad, for which contracting its three ground vertices results in the known complete graph, K4.

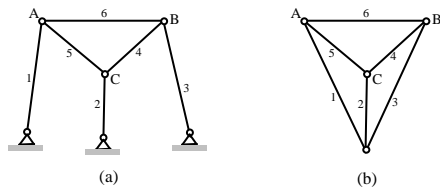


Figure 14. Transforming an Assur Graph into a generic cycle (contracted Assur Graph).
a) The triad. b) The corresponding contracted Assur Graph – K4.

It can be easily verified that all the Assur Graphs appearing in table 1 can be reformulated in the terminology of generic cycles. Thus, since for the latter it was proved that the two operations guarantee completeness and soundness thus it is valid also for all Assur Graphs.

Now that we have all the primitive building blocks, the AGs, we have the ability to construct all the

different topologies of planar linkages, as highlighted in the following section.

4. TOPOLOGICAL SYNTHESIS OF ALL 2D MECHANISMS THROUGH ASSUR GRAPHS

Once we have all the AGs in 2D it is possible to derive all the topological data of all the 2D mechanisms, by composing different AGs under the condition that the following composition rule is satisfied:

The composition rule of AGs:

Let G and T be two AGs. G can be composed on T if:

- Any one of the ground vertices of G is connected to an inner vertex of T or to the ground.
- The number of vertices that G is connected to is greater than or equal to two.

Figure 15 depicts several examples of determinate trusses that are compositions of one triad and one dyad.

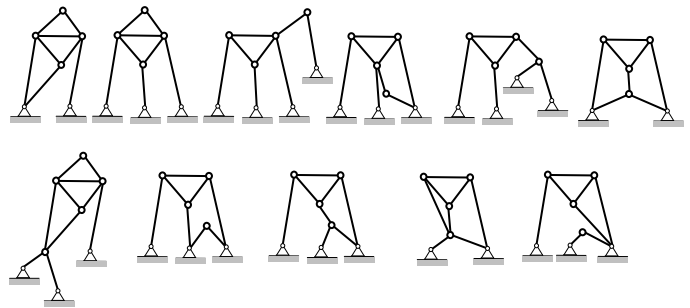


Figure 15. The composition resultants of a triad and a dyad.

Once we have all the compositions of Assur Graphs, i.e., various determinate trusses, the process of obtaining various linkages from them is done easily by connecting a driving link to one of the ground vertices. For example, in Figure 16 for a determinate truss (16a), which consists of a composition of a tetrad on a triad, there are four corresponding linkages since there are four ground vertices.

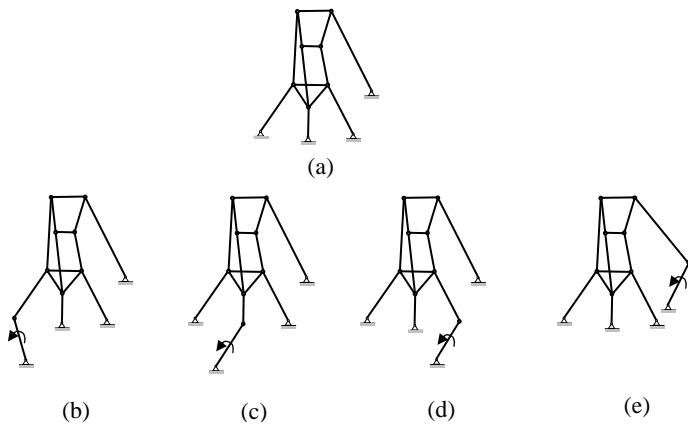


Figure 16 - Example of different linkages corresponding to a given determinate truss.

5. TOPOLOGICAL SYNTHESIS OF 3D MECHANISMS THROUGH ASSUR GRAPHS

The idea introduced above for 2D linkages was found to be applicable also for 3D. We start again with the dyad, this time consisting of three ground edges instead of two, as shown in Figure 17a. In the fundamental extension we replace each ground edge with a triangle, but this time two edges come out from each of the two new vertices, as shown in Figures 17b,c where a 3D triad was constructed through a fundamental extension operation from the 3D dyad. The ground edge that is replaced with the triangle and four ground edges are indicated by the bold edge.

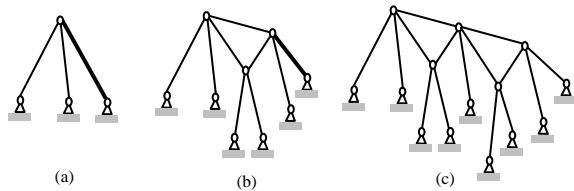


Figure 17. Example of a fundamental extension in 3D.
a) The 3D dyad. b) The 3D triad.

The idea of the classes, and that each fundamental Assur Graph is the representative of a class of Assur Graphs is the same as in 2D, only this time the operation of transforming one Assur Graph into the successor is done through 2-extension. In the 2-extension the vertex that is added is connected this time to two other vertices and not to one as is done in the 1-extension. Example of deriving an Assur Graph from 3D triad (Figure 18a) appears in Figure 18b,c where the edges being split are indicated by the bold edges.

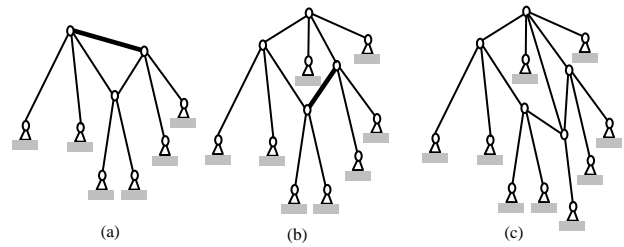


Figure 18. Example of applying the 2-extension in a 3D Assur Graph.
a) The fundamental Assur Graphs spatial triad.
b,c) The resulting Assur Graph after applying the 2-extension.

The structure of the 3D map is the same as for the 2D Assur Graph. The first column consists of the spatial dyad, which also does not have a correspondence in the 3D Assur Graph, thus there are no derivations in that column. The first row consists of the fundamental Assur Graphs, and each column contains all the derivations from that representative using 2-extension operation.

In contrast to 2D, there is no mathematical proof for the completeness of the 3D Assur Graphs. The main reason for that is that there are still mathematical problems that have not yet been resolved by the mathematicians in the rigidity theory community. Among these is the Assur Graph, appearing in Figure 19, for which there is no derivation from any fundamental Assur Graphs. The main problem is that the degree of each vertex in this graph is at least five.

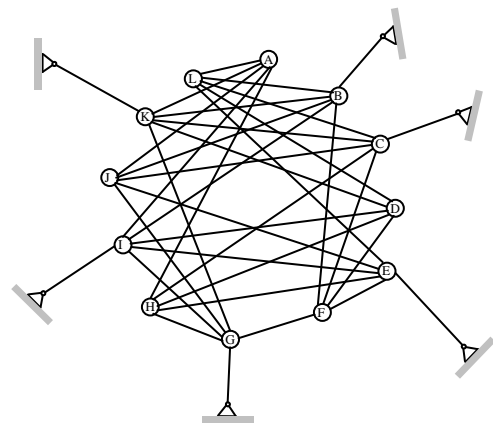


Figure 19. Example of a 3D Assur Graph for which there is no derivation from any fundamental Assur Graphs.

It is expected that some of the problems that engineers and mathematicians encounter in 3D are to

be solved by using Assur Graphs. For example, there are known structures and graphs for which all the formulas for calculating the degrees of freedoms, such as: Grubler equation or Laman's theorems provide wrong answers. However, when we decompose the graphs into Assur Graphs we reveal the reason for the problem and might be able to overcome it. For example, in Figure 20a appears a floating structure, a structure with no grounding, and although its DOF is equal to six, i.e., should be a rigid body, it has a finite motion since the upper and the lower parts can rotate related to each other along the virtual axis – FD.

But, when we first ground the structure by pinning joints E,H and G (Figure 20b) the resultant structure is decomposed into a triad and two dyads (Figures 20c,d). Now that we have the decomposition and the building blocks, the AGs, the problem is revealed. The triad is connected to the other structure, the two dyads, by only two joints. It is easy to prove (Shai, 2008) that in any dimension d , the number of joints that any AG should be connected to the remaining graph should be at least d .

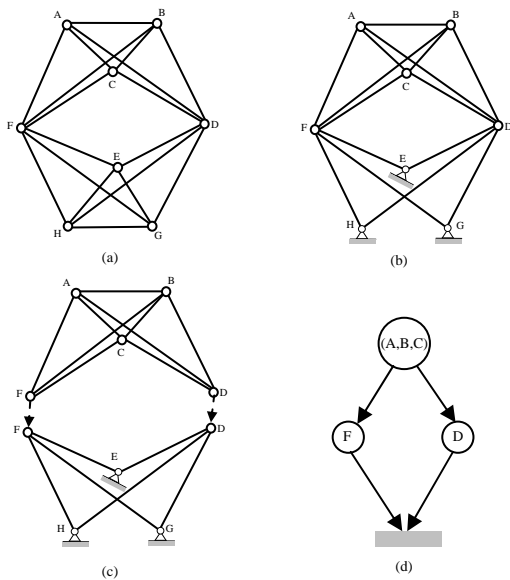


Figure 20 - The decomposition of the "double banana" structure into Assur Graphs
a) The floating structure. b) The corresponding determinate truss. c,d) the decomposition and its graph.

7. CONCLUSIONS AND FURTHER RESEARCH

The paper shows that the main criticism against Assur Groups: "Assur did not devise a convenient symbolic notation, and his classification system is

quite unwieldy to use" has been overcome. Moreover, it has been shown that there is an order in building all the topologies of the AGs, which all starts from the most basic topology - the dyad. Then through only two operations all the Assur Graphs are produced and arranged in an elegant order. The Assur Graphs are arranged in a table with infinite columns and infinite rows and each Assur graph in the table has a specific sequence of applying the mentioned two operations. It should be noted that it was proven in the references quoted that the two operations preserve the combinatorial properties of Assur Graphs.

Once we have all the Assur Graphs, when we connect several Assur graphs by connecting ground joints of one graph into inner joints of the other and adding driving links, we obtain the topology of all the linkages. Thus, we have the possibility to have all the topologies of all the planar linkages by applying different compositions on AGs and adding driving links.

Another limitation of the Assur Groups approach, as appears in the literature and mentioned in the paper, is that it is limited for planar mechanisms only. As appears in the paper and in previous publications of the author, it is possible to extend the AGs into three dimensions, but for now there are many AGs that it is impossible to derive them through known extensions. But, this paper reports for the first time in the engineering community that it is possible to solve known problems related to calculating degrees of freedoms of mechanisms through AGs.

A lot of effort is being put in our research group to establish the 3D map of all the AGs. For now, the map that we have is sound but not complete. All the graphs that are produced by the existing extension methods derive AGs but there are many AGs that cannot be derived. We all hope that at the next ASME conference we will be able to introduce the complete 3D map of the AGs, the extension of this paper.

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